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Diffusion of a Rectilinear Vortex in a Weakly Viscoelastic Liquid

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The classical problem of the decay of a rectilinear vortex because of the action of viscosity is extended to include the diffusion of vorticity in a weakly viscoelastic fluid. A second-order Rivlin-Ericksen stress tensor is used to model the viscoelastic nature of the flowfield; and the governing equations are solved by use of the Hankel transform. It is found that a "classical" vortex structure can exist in this type of fluid only if a certain constant in the stress tensor is positive, and, by recourse to kinetic theory, this constant is found to be negative for aqueous solutions of high molecular weight polymers.

Nomenclature

A	= Avogadro's number
$\overset{(1)}{B}{}^{ij}, \overset{(2)}{B}{}^{ij}$	= contravariant Rivlin-Ericksen acceleration tensors
C	= polymer concentration by weight
D/Dt	= $\partial/\partial t + \mathbf{u}\cdot\nabla$ = material derivative
$\hat{e}_r,\hat{e}_\theta,\hat{e}_z$	= unit vectors in cylindrical polar coordinates
$\mathbf{F}$	= surface force vector acting on a fluid particle
g_{ij}	= metric tensor
k	= Hankel transform variable
K	= Boltzmann's constant
M	= molecular weight of polymer
N	= number density of polymer
P	= pressure
T	= absolute temperature
t	= time
$\mathbf{u}$	= vector fluid velocity
u^i	= ith component of the contravariant velocity vector
x_1,x_2,x_3	= cylindrical polar coordinates corresponding to r, θ, z
β,γ	= material constants in the viscoelastic stress tensor
Γ	= circulation
δ_{ij}	= Kronecker delta ($\delta_{ij} = 0$, $i \neq j$, $\delta_{ii} = 1$)
μ	= absolute viscosity
$[\mu]$	= intrinsic viscosity
ν	= kinematic viscosity
ρ	= density
τ^{ij}	= contravariant stress tensor
τ_1	= relaxation time
$\boldsymbol{\omega}$	= vorticity vector

Subscripts

$(\ )_v$	= viscous variable
$(\ )_{ve}$	= viscoelastic variable

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Introduction

THE decay of a rectilinear vortex because of the action of viscosity is a problem of fundamental importance in classical hydrodynamics. The solution of this problem is extended here to include vortex diffusion in a weakly viscoelastic medium which is described by a second-order Rivlin-Ericksen stress tensor.

Interest in this problem is an outgrowth of the fact that recently much engineering effort has been expended in the study of dilute aqueous solutions of high molecular weight polymers. This class of fluids is important because of their ability to greatly reduce the drag of a submerged body in turbulent flow,[1] and because the presence of polymers in a laminar boundary layer tends to inhibit cavitation.[2]

Injection of such drag reducing additives into a boundary layer means that control surfaces and propulsers must function in weakly viscoelastic environments. All classical analyses of such things as propeller blades and stabilizing fins require a knowledge of the behavior of idealized vortices; hence it is of particular interest to understand the nature of vortex flow in this type of fluid.

The second-order Rivlin-Ericksen approximation to the stress tensor is used in this study for several reasons. Most important of these is that it probably represents the limit of tractability for an analytic investigation and because of this, it enjoys a wide application in the literature. In addition, it has been successful in demonstrating normal stress effects in steady laminar shearing flows of weakly viscoelastic fluids,[3] and has been used in a study of viscoelastic turbulence.[4]

The theoretical development of this problem includes a simultaneous treatment of vortex diffusion in both a weakly viscoelastic fluid and in a purely viscous fluid (which is seen to be a limiting case of the viscoelastic flow). This parallel treatment highlights the differences and similarities between the two flowfields and facilitates comparisons with classical results.

Mathematical Formulation

The equations governing this problem consist of the momentum balance

$$\rho D\mathbf{u}/Dt = -\nabla P + \mathbf{F} \tag{1}$$

and the incompressible continuity equation

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

In terms of a contravariant stress tensor τ^{ij}, the ith component of the surface force $\mathbf{F}$ is given by

$$F_i = \partial\tau^{ij}/\partial x_j \tag{3}$$

Geometrically, a rectilinear vortex is most easily described in cylindrical coordinates which are defined by

$$x_1 = r,\ x_2 = \theta,\ x_3 = z \tag{4}$$

where z is a Cartesian coordinate aligned with the axis of the vortex. The unit vectors associated with these coordinates are $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_z$.

In this coordinate system, the velocity field of a vortex has only one component which is azimuthal and which functionally depends only on r and t. Therefore

$$\mathbf{u} = \hat{e}_\theta u(r,t) = ru^{(2)}(r,t) \tag{5}$$

where $u^{(2)}(r,t)$ is the contravariant azimuthal velocity. Consequently, the vorticity field is given by

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{u} = (\hat{e}_z/r)(\partial/\partial r)(ru) \tag{6}$$

and the corresponding distribution of circulation is

$$\Gamma = 2\pi \int_0^r \omega(r,t)rdr = 2\pi ru(r,t) \tag{7}$$

Using Eq. (5), the continuity equation is satisfied identically. The various terms in the momentum equation (with the exception of the surface force) become

$$Du/Dt = \hat{e}_\theta(\partial u/\partial t) - \hat{e}_r(u^2/r) \tag{8}$$

and

$$\nabla P = \hat{e}_r(\partial P/\partial r) \tag{9}$$

which reduces Eq. (1) to

$$\hat{e}_\theta(\partial u/\partial t) - \hat{e}_r(u^2/r) = -\hat{e}_r(\partial P/\partial r) + \mathbf{F} \tag{10}$$

In order to complete the mathematical formulation of the problem, it is necessary to determine the surface force $\mathbf{F}$ which requires specification of the stress tensor appearing in Eq. (3). For the purposes of this study, it shall be assumed that the viscoelastic nature of the fluid in question is adequately represented by a second-order Rivlin-Ericksen stress tensor which has the form

$$\tau^{ij} = \mu_{ve} \overset{(1)}{B}{}^{ij} + \beta \overset{(1)}{B}{}^{ik} g_{kl} \overset{(1)}{B}{}^{lj} + \gamma \overset{(2)}{B}{}^{ij} \tag{11}$$

Here μ_{ve} is the zero shear absolute viscosity, β and γ are material constants, and g_{ij} is a metric tensor. The $\overset{(N)}{B}{}^{ij}$ are the contravariant Rivlin-Ericksen acceleration tensors which, in generalized tensor notation,[5] are given by

$$\overset{(1)}{B}{}^{ij} = u^i{}_{,m}g^{mj} + u^j{}_{,m}g^{im} - 2g^{ij}u^m{}_{,m} \tag{12}$$

and

$$\overset{(2)}{B}{}^{ij} = (D/Dt)\overset{(1)}{B}{}^{ij} + 2\overset{(1)}{B}{}^{ij}u^m{}_{,m} - u^i{}_{,m}\overset{(1)}{B}{}^{mj} - u^j{}_{,m}\overset{(1)}{B}{}^{im} \tag{13}$$

In the previous u^k is the kth component of the contravariant velocity vector and $g^{ij} = (g_{ij})^{-1}$.

It should be noted, as was pointed out by Ubler[6] and others, that for steady laminar shearing flows expression (11) is an exact representation of the stress field provided μ_{ve}, β, and γ are taken as functionally dependent on $\text{tr}[\overset{(1)}{B}{}^{ij}]^2$. This assumes only that the stress is an isotropic, hereditary functional of the history of deformation. In this study, it shall be assumed that these material parameters are independent of strain rate.

In order to orient the reader to the meaning of these tensorial expressions, and to subsequent manipulations, it should be realized that in the limit $\gamma = \beta = 0$, expression (11) reduces to the familiar Navier-Stokes stress tensor. That is

$$\tau^{ij} = u_v \overset{(1)}{B}{}^{ij} \tag{14}$$

and

$$\mathbf{F}_v = \mu_v\nabla^2\mathbf{u}_v = \hat{e}_\theta\mu_v[(\partial^2 u_v/\partial r^2) + (1/r)(\partial u_v/\partial r) - (u_v/r^2)] \tag{15}$$

After substituting expressions (12) and (13) into expression (11), taking account of the fact that the divergence of velocity $u^m{}_{,m}$ is zero, the viscoelastic force vector is found to be

$$\mathbf{F}_{ve} = \hat{e}_\theta\mu_{ve}\left[\frac{\partial^2 u_{ve}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{ve}}{\partial r} - \frac{u_{ve}}{r^2}\right] + \hat{e}_r\beta\frac{\partial}{\partial r}\left[r^2\left(\frac{\partial}{\partial r}\frac{u_{ve}}{r}\right)^2\right] + \gamma\frac{D}{Dt}\hat{e}_\theta\left[\frac{\partial^2 u_{ve}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{ve}}{\partial r} - \frac{u_{ve}}{r^2}\right] + \gamma\hat{e}_r\left[\frac{\partial}{\partial r}\left(u_{ve}\frac{\partial}{\partial r}\frac{u_{ve}}{r}\right) + \frac{u_{ve}}{r}\frac{\partial}{\partial r}\frac{u_{ve}}{r} + r\left\{\left(\frac{\partial}{\partial r}\frac{u_{ve}}{r}\right)^2 + \frac{u_{ve}}{r^2}\frac{\partial}{\partial r}\frac{u_{ve}}{r}\right\}\right] \tag{16}$$

Thus, in component form, the momentum equation for vortex flow in a weakly viscoelastic fluid becomes

$$\rho\frac{\partial u_{ve}}{\partial t} = \left(\mu_{ve} + \gamma\frac{\partial}{\partial t}\right)\left[\frac{\partial^2 u_{ve}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{ve}}{\partial r} - \frac{u_{ve}}{r^2}\right] \tag{17}$$

and

$$\frac{\partial P_{ve}}{\partial r} = \rho\frac{u_{ve}^2}{r} + \beta\frac{\partial}{\partial r}\left[r^2\left(\frac{\partial}{\partial r}\frac{u_{ve}}{r}\right)^2\right] + \gamma\left[\frac{\partial}{\partial r}\left(u_{ve}\frac{\partial}{\partial r}\frac{u_{ve}}{r}\right) + \frac{u_{ve}}{r}\frac{\partial}{\partial r}\frac{u_{ve}}{r} + r\left\{\left(\frac{\partial}{\partial r}\frac{u_{ve}}{r}\right)^2 + \frac{u_{ve}}{r}\frac{\partial}{\partial r}\frac{u_{ve}}{r}\right\} - \frac{u_{ve}}{r}\left(\frac{\partial^2 u_{ve}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{ve}}{\partial r} - \frac{u_{ve}}{r^2}\right)\right] \tag{18}$$

The solution of this problem is somewhat simplified if Eq. (6) is used to transform Eq. (17) into a diffusion equation for vorticity. This transformation yields

$$\frac{\partial\omega_{ve}}{\partial t} = \nu_{ve}\left(1 + \frac{\gamma}{\mu_{ve}}\frac{\partial}{\partial t}\right)\left[\frac{\partial^2\omega_{ve}}{\partial r^2} + \frac{1}{r}\frac{\partial\omega_{ve}}{\partial r}\right] \tag{19}$$

This completes the mathematical formulation of the problem since a knowledge of the vorticity field allows all other variables to be determined by quadrature. Results applicable to the purely viscous case may be obtained from these, and subsequent expressions, by simply setting γ and β equal to zero.

Formal Solution

The Hankel transform provides an extremely instructive method for uniquely solving diffusion equations like Eq. (19)

in an unbounded domain. This transform and its inverse are defined by

$$\omega(k,t) = \int_0^\infty \omega(r,t)J_0(kr)rdr \tag{20}$$

and

$$\omega(r,t) = \int_0^\infty \omega(k,t)J_0(kr)kdk \tag{21}$$

Here $\omega(k,t)$ is the Hankel transform of $\omega(r,t)$ and k is the transform variable. J_0 is the zeroth-order Bessel function.

Taking the transform of Eq. (19) gives

$$\partial\omega_{ve}(k,t)/\partial t = -\nu_{ve}[1 + (\gamma/\mu_{ve})(\partial/\partial t)]k^2\omega_{ve}(kt) \tag{22}$$

The solution to this equation is

$$\omega_{ve}(k,t) = \omega_{ve}(k,0)\exp\{-\nu_{ve}k^2t/[1 + (\gamma/\mu_{ve})\nu_{ve}k^2]\} \tag{23}$$

where $\omega_{ve}(k,0)$ is a constant of integration and physically represents the Hankel transform of some initial distribution of vorticity.

Consequently, the solution for the vorticity field, if it exists, is obtained by using expression (21) to invert (23). Thus,

$$\omega_{ve}(r,t) = \int_0^\infty \omega_{ve}(k,0)\exp\left[-\frac{\nu_{ve}k^2t}{1 + (\gamma/\mu_{ve})\nu_{ve}k^2}\right]J_0(kr)kdk \tag{24}$$

The corresponding solution to the problem of the diffusion of vorticity in a purely viscous fluid is

$$\omega_v(r,t) = \int_0^\infty \omega_v(k,0)\exp[-\nu_vk^2t]J_0(kr)kdk \tag{25}$$

Similar expressions for the circulations and velocities may be obtained from Eq. (7) and are

$$\Gamma_{ve} = 2\pi ru_{ve} = 2\pi\int_0^\infty \omega_{ve}(k,0) \times \exp\left[-\frac{\nu_{ve}k^2t}{1 + (\gamma/\mu_{ve})\nu_{ve}k^2}\right]rJ_1(kr)dk \tag{26}$$

and

$$\Gamma_v = 2\pi ru_v = 2\pi\int_0^\infty \omega_v(k,0)\exp[-\nu_{ve}k^2t]rJ_1(kr)dk \tag{27}$$

where use has been made of the fact that

$$\int_0^r J_0(kr)krdr = rJ_1(kr) \tag{28}$$

Lastly, the determination of the pressure fields, while more complicated, follows directly from Eq. (18).

The previous expressions represent only formal solutions since the conditions for which they are valid have not been determined. Any distribution of vorticity which has a time history corresponding to the conventionally accepted kinematics of a rectilinear vortex must be described by expression (24). The existence of such a solution depends on the existence of the integral in expression (24) which, in turn, depends upon the nature of the initial distribution of vorticity and upon the sign of the various material constants. These considerations apply equally to the purely viscous case and to expression (25).

Therefore, in order to determine the relevance of these formal solutions, it is necessary to examine the classical diffusion of a rectilinear vortex in a purely viscous fluid and also to examine the nature of the material constants appearing in the second-order Rivlin-Ericksen stress tensor.

Discussion of the Formal Solutions and of the Nature of a "Classical" Rectilinear Vortex

The existence of the integral in Eq. (25), which represents the formal solution for the one-dimensional diffusion of vorticity in a purely viscous fluid, poses no problem. For a fixed t, the integrand is a well behaved function of k and r and, provided $\omega_v(k,0)$ is bounded, it is exponentially decaying for k tending to infinity.

In the classical treatment of the dissolution of a vortex because of the action of viscosity, it is usually assumed that initially the vortex is a vortex line. That is, the vorticity, at the initial instant, is given by a delta function implying a finite circulation but no spatial extent of the vortex core. For this case,

$$\omega_v(k,0) = \Gamma_0/\pi \tag{29}$$

where Γ_0 is the magnitude of the initial circulation; and expression (25) integrates to yield the classical expression for vortex diffusion given by

$$\omega_v(r,t) = (\Gamma_0/\pi t)\exp(-r^2/4\nu_vt) \tag{30}$$

More generally, a viscous rectilinear vortex may be defined as any spatially confined, unidirectional distribution of vorticity describable by expression (25). From the previous discussion, such a structure can exist in a purely viscous fluid provided its initial distribution of vorticity has a Hankel transform that is of bounded variation in k.

A "classical" vortex may then be defined as any vortex whose initial distribution of vorticity is normal; that is, whose initial distribution of vorticity is obtainable as an instantaneous snap shot of the vortex described by Eq. (30). In many ways, this specification of an initial condition is more representative of naturally occurring vortices than is the mathematical fiction of an initial delta function vorticity distribution.

Attention may now be focused on the existence of the integral in expression (24) representing the formal solution for the diffusion of vorticity in a second-order Rivlin-Ericksen fluid. The existence of this integral is drastically affected by the sign of the material constant γ. Consider first the case γ positive.

With this restriction, the exponential in the integrand tends to a constant as k tends to infinity; and, as a consequence, expression (24) does not converge for $\omega_{ve}(k,0)$ equal to a constant. However, specifying that the initial distribution is "classical" is more than sufficient to ensure the existence of a valid solution.

For the case γ negative, a bounded, nontrivial solution does not exist, regardless of the form of the initial vorticity distribution, since the integrand of expression (24) possesses a nonintegrable singularity at

$$k^2 = \mu_{ve}/\gamma\nu_{ve} = \rho/\gamma \tag{31}$$

This result is in agreement with the conclusions of Coleman et al.[7] who discussed instability, uniqueness, and nonexistence theorems for the equation

$$\partial V/\partial t = (\partial^2V/\partial x^2) - (\partial^3V/\partial x\partial t\partial x) \tag{32}$$

The physical problem which motivated these authors was the one-dimensional flow of a simple, incompressible, viscoelastic fluid with fading memory in which

$$V_x = 0,\ V_y = V(x,t),\ V_z = 0 \tag{33}$$

Bounding surfaces were located at $x = 0$ and $x = h$; and a second-order fluid model was assumed with γ negative. This is the bounded Cartesian analogue of the unbounded cylindrical problem described by Eq. (19). The pertinent results of their investigation are, that if the previous kinematics are assumed for $t < 0$, and if at $t = 0$ the boundary conditions

$$V(0,t) = V(h,t) = 0 \tag{34}$$

are imposed, while $V(x,0)$ is nonzero for some x, then, depending on the magnitude of h, the velocity must become arbitrarily large as $t \to \infty$, or no flow of the type of Eqs. (33) is possible for $t > 0$. They conclude from this that either the second-order stress tensor is not valid for unsteady shearing flows, or a flow exists which does not conform to the assumed kinematics.

In like manner assuming that the viscoelastic nature of a particular fluid is adequately represented by a second-order Rivlin-Ericksen stress tensor implies that a "classical" vortex can exist in that fluid only if γ is positive. If γ is negative it is impossible for any physically meaningful initial distribution of vorticity to have a subsequent time history that corresponds to the conventionally accepted kinematics of vortex flow.

Case of Dilute Aqueous Solutions of High Molecular Weight Polymers

Dilute aqueous solutions of high molecular weight polymers comprise an extremely important class of weakly viscoelastic fluids because of their ability to reduce the hydrodynamic drag of marine vehicles. Therefore, it is useful to examine the nature of the constants in the second-order Rivlin-Ericksen stress tensor for these fluids. Unfortunately, very little pertinent experimental data are available; however, general expressions for these constants may be obtained using the results of kinetic theory. The following presentation closely parallels that of Singh[4] but several errors in his paper are corrected here.[8]

Rivlin[9] calculated the additional momentum transfer because of the presence of polymer molecules in a solvent. From his work, the constant β (referred to as the normal stress coefficient) is found to be

$$\beta = 2(\mu_{ve} - \mu_v)^2/NKT \tag{35}$$

where K is Boltzmann's constant, T is the absolute temperature, and N is the number of polymer molecules per unit volume.

Rouse[10] calculated a series of relaxation times for a polymer molecule in solution. The largest of these relaxation times is related to γ by

$$\gamma = -NKT\tau_1^2 \tag{36}$$

In the previous equation,

$$\tau_1 = 6(\mu_{ve} - \mu_v)/\pi^2 NKT \tag{37}$$

so

$$\gamma = -36(\mu_{ve} - \mu_v)^2/\pi^4 NKT \tag{38}$$

These expressions may be put in a more useful form by eliminating the number density N in terms of the concentration by weight C, Avogadro's number A, and the molecular weight M. Also, the viscosity difference may be expressed in terms of the intrinsic viscosity of the solution $[\mu]$. Thus,

$$\beta = (2MC/5AKT)\mu_v^2[\mu]^2 \tag{39}$$

and

$$\gamma = -(36MC/\pi^4 AKT)\mu_v^2[\mu]^2 \tag{40}$$

Coleman and Markovitz[3] also arrive at the conclusion that γ is negative but by different reasoning. They discuss the case of a simple fluid with fading memory for which the second-order fluid model is shown to be correct to orders greater than one in the relaxation time scale. This applies provided either the fluid is weakly viscoelastic or the deformations are sufficiently slow. They find that γ, as well as the zero shear viscosity μ_v, are related to the stress relaxation function of classical, linear viscoelasticity by

$$\mu_v = \int_0^\infty G(s)ds \tag{41}$$

and

$$\gamma = -\int_0^\infty sG(s)ds \tag{42}$$

Here s is the difference between the present instant of time and some previous time during the history of the deformation. G is a characteristic function that gives the ratio of shear stress to shear strain for a material that experiences a step change in loading. The stress relaxation function is obviously positive which implies μ_v is positive, as it should be, and therefore γ is negative.

Experimentally these results have been verified by Markovitz and Brown[11] for solutions of polyisobutylene in cetane. Their results give β = 1.0 g/cm and γ = −0.2g/cm for a 5.4% solution at 30°C.

As a consequence, a "classical" vortex structure cannot exist in this important class of engineering fluids if it is assumed that their viscoelastic behavior is adequately represented by a second-order Rivlin-Ericksen stress tensor.

Conclusions

The results of this investigation raise some interesting questions concerning the general utility of the second-order Rivlin-Ericksen approximation. While it is difficult to make any definitive statements in the absence of supportive experimental data, one would certainly expect to encounter vortices in dilute aqueous polymer solutions. If this is true, then the second-order fluid model fails to give physical results when applied to this simple unsteady flowfield. In light of this, its application to unsteady flow problems in general, and to the problem of polymer turbulence in particular would seem questionable.

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